Kolmogorov systems under the telegraph noise

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Faculty of Mathematics, Mechanics and Informatics,
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Talk at CoMMISCo'2010

Paris, France 11/10/2010

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Introduction

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$$\begin{cases} \dot{x} &= xf(x,y) \\ \dot{y} &= yg(x,y) \end{cases} \tag{1}$$

  where x and y represent the population density and $f(x,y)$, $g(x,y)$ are the capita growth rate of each species. Usually, such systems are called Kolmogorov systems.
- An important topic is whether every orbit starting in the interior of the first quadrant of the phase plane (i.e. $x(0) > 0, y(0) > 0$) remains persistent.

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- There have been numerous works investigating the dynamics of the positive solutions such as the uniformly strong persistence, the extinction and ultimately boundedness of the System (1).
- Almost assume that species live in a constant environment, i.e., the capita growth rates $f(x, y)$ and $g(x, y)$ are deterministic functions.
- It is clear that it is not the case in reality and that it is important to take into account the variability of the environment. The variability of the environment may be expressed under the random factors

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- In the simplest case, one might consider that environmental conditions can switch randomly between two states, for instance: hot state and cold one, dry state and wet one... Thus, we can suppose there is a telegraph noise ξ_t affecting on the model in the form of switching between two-element set, $E = \{+, -\}$. With different states, the dynamics of model are different. The stochastic displacement of environmental conditions provokes model to change from the system in state $+$ to the system in state $-$ and vice versa. Since ξ_t takes only two values, we call it a telegraph noise.
- In this case, the mathematical model can be described by the random Kolmogorov equation

$$\begin{cases} \dot{x}(t) &= xa(\xi_t, x, y) \\ \dot{y}(t) &= yb(\xi_t, x, y) \end{cases} \tag{2}$$

where, $a(\pm, x, y), b(\pm, x, y)$ defined on $E \times \mathbb{R}^2_+$, valued in $\mathbb{R}$ are continously differentiable in $(x, y) \in \mathbb{R}^2_+$ where

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Pathwise dynamic behavior of classical competition systems under telegraph noise

- The classical competition model of (2) has the form

$$\begin{cases} \dot{x}(t) = x(a(\xi_t) - b(\xi_t)x - c(\xi_t)y) \\ \dot{y}(t) = y(d(\xi_t) - e(\xi_t)x - f(\xi_t)y) \end{cases}, \tag{5}$$

  where $a(\pm), b(\pm), c(\pm), d(\pm), e(\pm), f(\pm)$ are positive constants.
- The noise ξ_t switchs (5) between the deterministic Lotka-Volterra system

$$\begin{cases} \dot{x}_+(t) = x_+(a(+) - b(+)x_+ - c(+)y_+) \\ \dot{y}_+(t) = y_+(d(+) - e(+)x_+ - f(+)y_+), \end{cases} \tag{6}$$

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Suppose that (ξ_t) has the transition intensities $+ \overset{\alpha}{\to} -$ and $- \overset{\beta}{\to} +$ with $\alpha > 0, \beta > 0$.

- The process (ξ_t) has a unique stationary distribution

$$p = \lim_{t\to\infty} \mathbb{P}\{\xi_t = +\} = \frac{\beta}{\alpha+\beta};\ q = \lim_{t\to\infty} \mathbb{P}\{\xi_t = -\} = \frac{\alpha}{\alpha+\beta}.$$

- The trajectories of (ξ_t) are piecewise-constant, cadlag functions. Let

$$0 = \tau_0 < \tau_1 < \tau_2 < ... < \tau_n < ...$$

be its jump times. Put

$$\sigma_1 = \tau_1 - \tau_0,\ \sigma_2 = \tau_2 - \tau_1, ..., \sigma_n = \tau_n - \tau_{n-1} ...$$

- $\sigma_1 = \tau_1$ is the first exile from the initial state, σ_2 is the time the process (ξ_t) spends in the state into which it moves from the first state... It is known that $(\sigma_k)_{k=1}^{\infty}$ are independent in the condition of given sequence $(\xi_{\tau_k})_{k=1}^{\infty}$. Note that $(\sigma_k)_{n=1}^{\infty}$ is a sequence of conditionally independent random variables, valued in $[0, \infty)$. Moreover, if $\xi_0 = +$ then σ_{2n+1} has the exponential density

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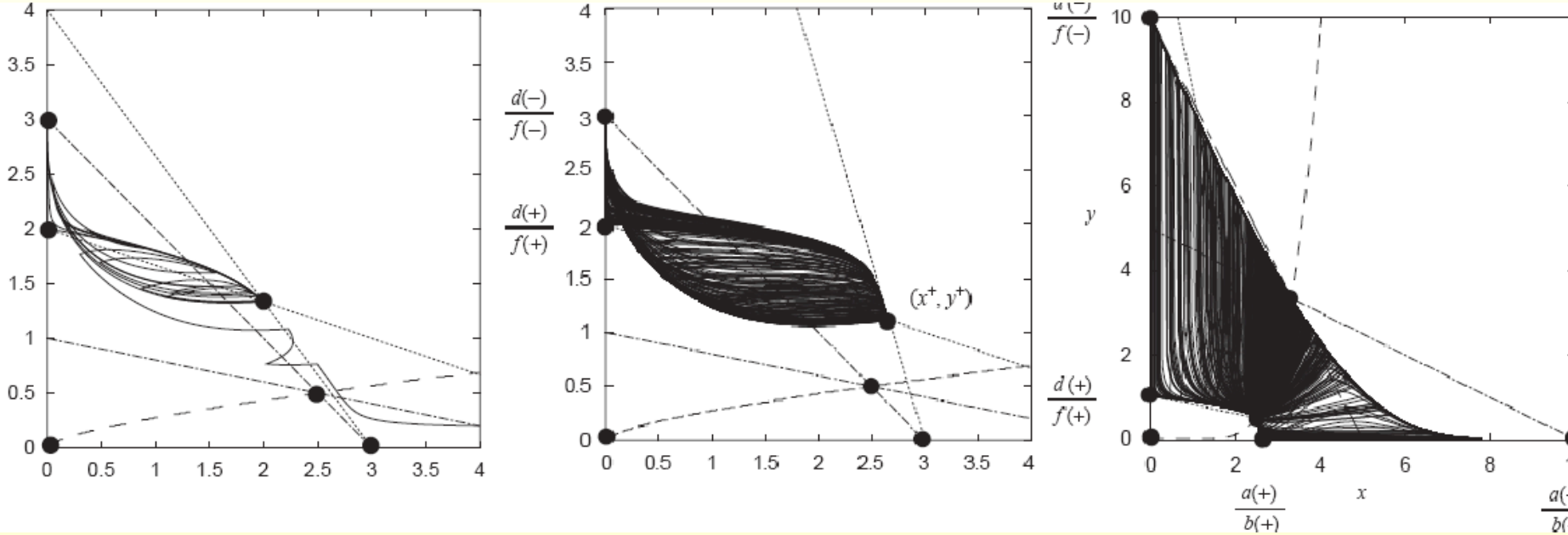
The simulation result shows the dynamic behavior of (5) is



The purpose of this talk is to discuss about this situation:

- The existence of an attractor
- The existence of stationary distributions
- Properties of this distributions

Now we turn to the general system (1). We suppose that the system (1) is a competion-type system. This means that the following assumptions are given

Assumption 2.1

1. $\dfrac{\partial a(\pm, x, 0)}{\partial x} < 0 \; \forall x > 0.$
2. $a(\pm, 0, 0) > 0, \lim\limits_{x \to \infty} a(\pm, x, 0) < 0.$
3. $\dfrac{\partial b(\pm, 0, y)}{\partial y} < 0 \; \forall y > 0.$
4. $b(\pm, 0, 0) > 0, \lim\limits_{y \to \infty} b(\pm, 0, y) < 0.$

Assumption 2.2

There is a compact set $D \subset \mathbb{R}^2_+$ such that D is an invariant set for both the systems (3) and (4). Moreover, for all $(x, y) \in \mathbb{R}^2_+$, there is $T \geq 0$ such that $(x_1(t), y_1(t)) \in D$, $(x_2(t), y_2(t)) \in D \; \forall t > T$

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Lemma 2.1

Suppose that the system

$$\begin{cases} \dot{x}(t) = f(x, y) \\ \dot{y}(t) = g(x, y), \end{cases}$$

where $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ has an equilibrium (x^*, y^*) to be globally asymptotically stable, i.e., (x^*, y^*) is stable and every solution $(x(t), y(t))$ defined on $[0, \infty)$ satisfies $\lim_{t\to\infty}(x(t), y(t)) = (x^*, y^*)$. Then, for any compact set $\mathcal{K} \subset \mathbb{R}^2$, for any neighborhood U of (x^*, y^*), there exists a number $T^* > 0$ such that $(x(t, x_0, y_0), y(t, x_0, y_0)) \in U$ for any $t > T^*$, provided $(x_0, y_0) \in \mathcal{K}$.

- We define the (random) $\omega-$limit set of the trajectories starting in a closed set B

$$\Omega(B,\omega) = \bigcap_{T>0} \bigcup_{t>T} \overline{\big(x(t,\cdot,\omega), y(t,\cdot,\omega)\big)B}.$$

In particular, the $\omega-$limit set of the trajectory starting from an initial value (x_0, y_0) is

$$\Omega(x_0, y_0, \omega) = \bigcap_{T>0} \bigcup_{t>T} \overline{\big(x(t, x_0, y_0, \omega), y(t, x_0, y_0, \omega)\big)}.$$

- Our task is to describe the $\omega-$limit set of the trajectories of (1).
- With the above assumption, if $u(t)$ is the solution of the system on the boundary

$$\dot{u}(t) = u(t)a(\xi_t, u(t), 0), \quad u(0) \in [0, \infty), \tag{8}$$

- then the process $(\xi_t, u(t))$ has a unique stationary distribution with the density (μ^+, μ^-) which is easyly computed

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Put

$$\begin{aligned}\lambda_1 &= \int_{[v^+,v^-]} (pa(+,v,0)\nu^+(v) + qa(-,v,0)\nu^-(v))dv.\\ \lambda_2 &= \int_{[u^+,u^-]} (pb(-,u,0)\mu^+(u) + qb(-,u,0)\mu^-(u))du.\end{aligned} \tag{10}$$

The following theorem says that $x(t)$ and $y(t)$ are weakly persistent:

Theorem

For any $x_0 > 0, y_0 > 0$.

a) *If $\lambda_1 > 0$ then there exists a $\delta_1 > 0$ such that* $\limsup_{t\to\infty} x(t, x_0, y_0) > \delta_1$, *a.s.*

b) *In case $\lambda_2 > 0$ there exists a $\delta_2 > 0$ such that* $\limsup_{t\to\infty} y(t, x_0, y_0) > \delta_2$ *a.s.*

However, this theorem does not say that there is a t such that $(x(t), y(t))$ far from boundary. But we have

From now on, we suppose that $\lambda_1 > 0, \lambda_2 > 0$.

Lemma 2.2

There are infinitely many $s_n > 0$ such that $s_n > s_{n-1}$, $\lim_{n\to\infty} s_n = \infty$ and $x(s_n) \geq \delta, y(s_n) \geq \delta$ for all $n \in \mathbb{N}$.

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To "control" the trajectories, we need one of deterministic system (either (2) or (3)) to be asymptotically stable:

Assumption 2.3

On the quadrant $\operatorname{int}\mathbb{R}^2_+$, both systems (3), (or (4)) has the globally stable positive states (x^*_+, y^*_+), respectively.

By this assumption we get

Lemma 2.4

Let Assumption 2.3 be satisfied. Then, with probability 1, there are infinitely many $s_n = s_n(\omega) > 0$ such that $s_n > s_{n-1}$, $\lim_{n\to\infty} s_n = \infty$ and $x(s_n) \geq \delta, y(s_n) \geq \delta$ for all $n \in \mathbb{N}$.

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To "control" the trajectories, we need one of deterministic system (either (2) or (3)) to be asymptotically stable:

Assumption 2.3

On the quadrant $\operatorname{int}\mathbb{R}^2_+$, both systems (3), (or (4)) has the globally stable positive states (x^*_+, y^*_+), respectively.

By this assumption we get

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The inequalities $x(s_n) \geq \delta, y(s_n) \geq \delta$ is not enough, we have to show that they happen when s_n is switching times of x_t.

Lemma 2.4

Let Assumption 2.4 be satisfied. Then, for δ mentioned above , with probability 1, there are infinitely many $k = k(\omega) \in \mathbb{N}$ such that $x_{2k+1} > \sigma(\sigma(\delta)), y_{2k+1} > \sigma(\sigma(\delta))$.

We describe the $\omega-$limit set. Put

$$S = \left\{(x, y) = \pi_{t_n}^{\varrho(n)} \cdots \pi_{t_1}^{\varrho(1)}(x_+^*, y_+^*) : 0 < t_1 < t_2 < \cdots < t_n;\ n \in \mathbb{N}\right\}. \tag{11}$$

where $\varrho(k) = (-1)^k$.

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Theorem 2.2

Suppose that (3) has a globally stable equilibrium (x_+^*, y_+^*) Assumption 2.4 is satisfied. Then

a) The closure $\bar{S}$ of S is a subset of the ω-limit set $\omega(x_0, y_0)$.

b) If there exists a $t_0 > 0$ such that the point $(\bar{x}_0, \bar{y}_0) = \pi_{t_0}^-(x_+^*, y_+^*)$ satisfying the following condition

$$\det \begin{pmatrix} a(+, \bar{x}_0, \bar{y}_0) & a(-, \bar{x}_0, \bar{y}_0) \\ b(+, \bar{x}_0, \bar{y}_0) & b(-, \bar{x}_0, \bar{y}_0) \end{pmatrix} \neq 0. \tag{12}$$

Then, the closure $\bar{S}$ of S is the ω-limit set $\omega(x_0, y_0)$ as well as the attractor of all the solutions starting in $\mathrm{int}\mathbb{R}_+^2$ with probability 1. Moreover, $\bar{S}$ absorbs all positive solutions in the sense that for any initial value $(x_0, y_0) \in \mathrm{int}\mathbb{R}_+^2$, the value $\gamma(\omega) = \inf\{t > 0 : (x(s, x_0, y_0, \omega), y(s, x_0, y_0, \omega)) \in \bar{S}\, \forall\, s > t\}$ is finite outside a $\mathbb{P}$-null set.

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Remark

The assumption there exists a $t_0 > 0$ such that the point $(\bar{x}_0, \bar{y}_0) = \pi^-_{t_0}(x^*_+, y^*_+)$ satisfying (16) is equivalent to the condition: there exists a point $(x, y) \in \{\pi^-_t(x^*_+, y^*_+) : t \geq 0\}$ such that the curve $\{\pi^+_t(x, y) : t \geq 0\}$ is not contained in $\{\pi^-_t(x^*_+, y^*_+) : t \geq 0\}$.

The semigroup and the stability in distribution

The process $z(t) = (x(t), y(t))$ is a homogeneous Markov process with the state space $\mathcal{V} := E \times \text{int}\mathbb{R}^2_+$ with the transition probability $P(t, i, z, B)$. Let λ be the Lebesgue measure on $\text{int}\mathbb{R}^2_+$; ℓ be the measure on E given by $\ell(+) = p, \ell(-) = q$. Denote by $m = \ell \times \lambda$. Let $\{P(t)\}_{t\geq 0}$ be the semigroup defined on the set of measures $\mathcal{P}(\mathcal{V})$ given by

$$P(t)\nu(B) = \int P(t, i, z, B)\nu(di, dz), \quad \nu \in \mathcal{V},\ B \in \mathcal{B}(\mathcal{V}).$$

It is clear that if ν is the distribution of $(\xi_0, z(0))$ then $P(t)\nu$ is the distribution of $(\xi_t, z(t))$.

lemma 3.1

If ν^* is a stationary distribution of the process $(\xi_t, z(t))$, i.e., $P(t)\nu^* = \nu^*$ for all $t \geq 0$ with $\nu^*(E \times \text{int}\,\mathbb{R}^2_+) = 1$, then ν^* has the density f^* with respect to m and $\text{supp}(f^*) = E \times S$.

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Remark

The question whenever ν exists is still open. In our opinion if $\lambda_1 > 0$, $\lambda_2 > 0$, such a ν esists.

Theorem 3.1

Let the hypothesis of Theorem 2.2 be satisfied. If the semigroup $\{P(t)\}_{t\geq 0}$ has an invariant probability measure ν^*, then ν^* has the density f^* and the semigroup is asymptotically stable in the sense that $\lim_{t\to\infty} \|P(t)f - f^*\| = 0 \, \forall f \in \mathcal{D}$, where f is the initial distribution of $(x(0), \xi_0)$.

Applications

- We now apply the results studied in above sections to study the classical competition model (5)

$$\begin{cases} \dot{x}(t) = x(a(\xi_t) - b(\xi_t)x - c(\xi_t)y) \\ \dot{y}(t) = y(d(\xi_t) - e(\xi_t)x - f(\xi_t)y) \end{cases},$$

- With asumption that $b(\pm)f(\pm) - c(\pm)e(\pm) \neq 0$, the system (6) (resp. (7)) has the equilibrium (x_+^*, y_+^*) (resp. (x_-^*, y_-^*) where

$$x_\pm^* = \frac{a(\pm)f(\pm) - c(\pm)d(\pm)}{b(\pm)f(\pm) - c(\pm)e(\pm)}, \quad y_\pm^* = \frac{b(\pm)d(\pm) - a(\pm)e(\pm)}{b(\pm)f(\pm) - c(\pm)e(\pm)}.$$

  We choose $D = \{(x, y), 0 \leq x, y \leq M\}$ for $M > \max_{i \in E} \max\{\frac{a(i)}{b(i)}, \frac{a(i)}{c(i)}, \frac{d(i)}{e(i)}, \frac{d(i)}{f(i)}\}$.
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The values of λ is given by

$$\lambda_1 = \left(a(+) - \frac{c(+)}{f(+)}d(+)\right)p + \left(a(-) - \frac{c(-)}{f(-)}d(-)\right)q$$

$$+\frac{p}{\zeta}\left(\frac{c(+)}{f(+)} - \frac{c(-)}{f(-)}\right)\mathrm{sign}\left(\frac{d(+)}{f(+)} - \frac{d(-)}{f(-)}\right)\int_{v_1}^{v_2} \frac{|d(+) - f(+)v|^{\frac{\alpha}{d(+)}}|d(-) - f(-)v|}{v^{\frac{\alpha}{d(+)} + \frac{\beta}{d(-)} + 1}}$$

and

$$\lambda_2 = \left(d(+) - \frac{e(+)}{b(+)}a(+)\right)p + \left(d(-) - \frac{e(-)}{b(-)}a(-)\right)q$$

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We consider some typical cases

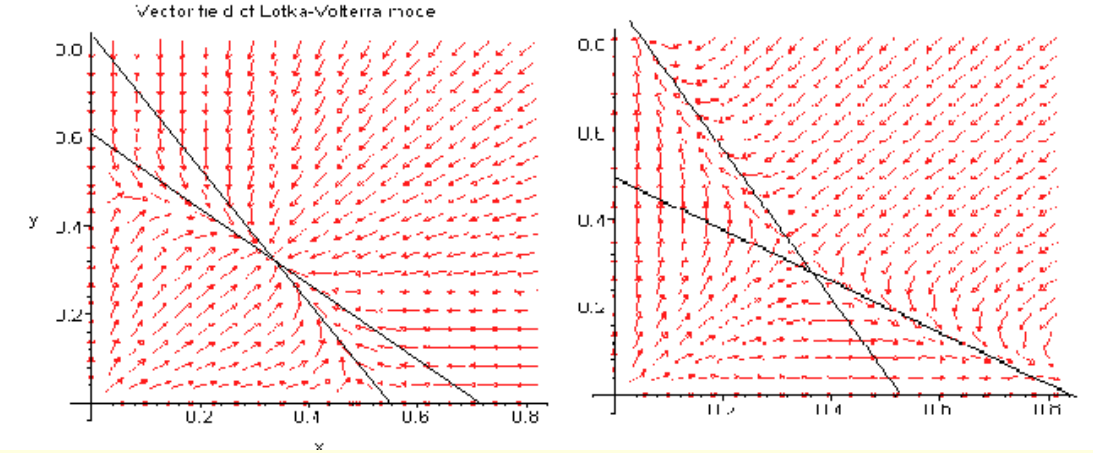


Figure: Orbit of the system, number of switching $n = 1000$.

- Both two systems (6) and (7) is stable: we hope that (7) is permanent.
- The system (6) is stable and (7) is bistable
- None of systems (6) and (7) has positive equilibrium point

Systems (6) and (7) are globally asymptotically stable

Suppose that

$$\frac{d(i)}{f(i)} < \frac{a(i)}{c(i)}; \quad \frac{d(i)}{e(i)} > \frac{a(i)}{b(i)} \text{ for } i \in E. \tag{13}$$

In this case, the systems (6) and (7) are globally asymptotically stable with the equilibrium $(x_{\pm}^*, y_{\pm}^*)$ and there exists a $(\bar{x}_0, \bar{y}_0) = \pi_{t_0}^-(x_+^*, y_+^*)$, $t_0 > 0$ such that (16) is satisfied.
Although two equilibriums are asymptotically stable, but we can give an example with $\lambda_1 < 0$ and $lim_{t\to\infty} y(t) = 0$. Also we can give an example where $\liminf_{t\to infty} y(t) = 0$ but there is a stationary distribution.
We illustrate this case an example where $a(+) = 1, b(+) = 1, c(+) = 0.1, d(+) = 7, e(+) = 6, f(+) = 1, a(-) = 3, b(-) = 1, c(-) = 0.1, d(-) = 7, e(-) = 2, f(-) = 1, x(0) = 2, y(0) = 3,$
Case A: $\alpha = 3, \beta = 4,\ \lambda_1 \approx 1.157;\ \lambda_2 \approx -0.545$
Case B: $\alpha = 0.3, \beta = 0.4,\ \lambda_1 \approx 1.157;\ \lambda_2 \approx 0.461$

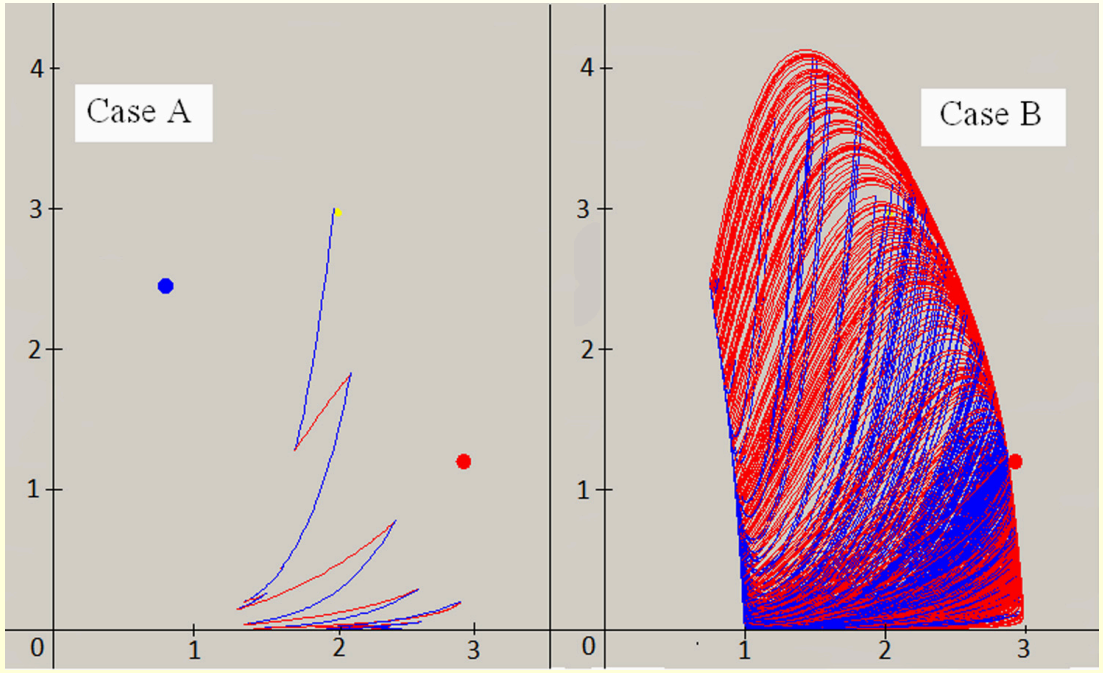


Figure: Orbit of the system, number of switching $n = 1000$.

We can add some further assumption to have

Suppose that the relations (13) hold and that either $\max_{i\in E}\left\{\frac{a(i)}{b(i)}\right\} < \min_{i\in E}\left\{\frac{d(i)}{e(i)}\right\}$ or $\max_{i\in E}\left\{\frac{d(i)}{f(i)}\right\} < \min_{i\in E}\left\{\frac{a(i)}{c(i)}\right\}$. The followings hold

a) If $(x_+^*, y_+^*) \neq (x_-^*, y_-^*)$, there exists uniquely a stationary distribution for the Markov process $(\xi_t, x(t), y(t))$. This stationary distribution has a density f^* with respect to the measure m on $E \times \mathbb{R}_+^2$. Further, this stationary density is asymptotically stable, i.e., $\lim_{t\to\infty} \|P(t)f - f^*\| = 0 \, \forall f \in L^1, \|f\| = 1$.

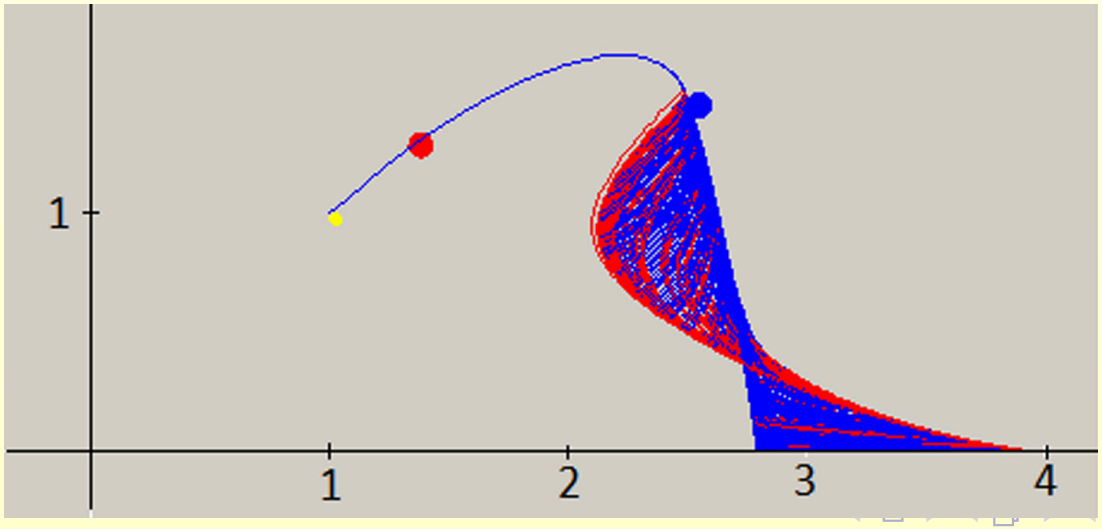
b) If $(x_+^*, y_+^*) = (x_-^*, y_-^*) = (x^*, y^*)$, then $\lim_{t\to\infty}(x(t), y(t) = (x^*, y^*)$ almost surely for any initial value $(x_0, y_0) \in \mathbb{R}_+^2$.

Systems (6) is globally asymptotically stable and (7) is bistable

Suppose that

$$\frac{d(+)}{f(+)} < \frac{a(+)}{c(+)}; \quad \frac{d(+)}{e(+)} > \frac{a(+)}{b(+)} \quad \text{but} \quad \frac{d(-)}{f(-)} > \frac{a(-)}{c(-)}; \quad \frac{d(-)}{e(-)} < \frac{a(-)}{b(-)}.$$

In this case, the system (6) has a unique positive stable state (x_+^*, y_+^*); the system (7) is bistable.



It is difficult to describe precisely the ω-limit sets of positive solution in the case where none of them has globally stable positive equilibrium. Note that the positivity of λ_i does not imply the existence of positive equilibrium of two deterministic systems. Consider an example where all positive solutions of (6) tend to $(0, 4)$ meanwhile those of (7) tend to $(3, 0)$ but $\lambda_1 \approx 0.5 > 0$, $\lambda_2 \approx 0.346 > 0$. Dynamics of solutions are illustrated by Figure 5.

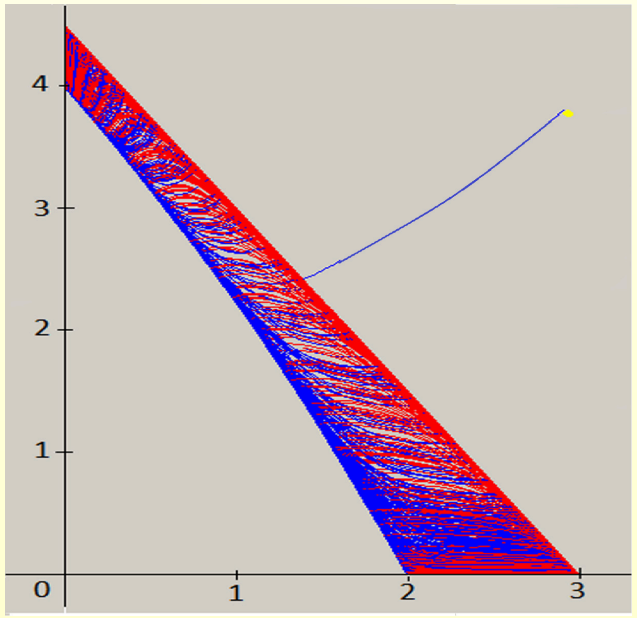


Figure: $a(+) = 6, b(+) = 3, c(+) = 2, d(+) = 12, e(+) = 4, f(+) =$

- By numerical solutions, we think that in case $\lambda_1 > 0, \lambda_2 > 0$, there exists uniquely a stationary density in $\mathrm{int}\mathbb{R}^2_+$. However, so far this is still an open question to us.
- Further, so far we still do not know the behavior of trajectories when either $\lambda_1 \leq 0$ or $\lambda_2 \leq 0$.

Pathwise dynamic behavior

Following the assumptions on the classical prey-predator we make the following assumptions

Assumption 3.1

1. $\dfrac{\partial a(\pm, x, 0)}{\partial x} < 0 \; \forall x \geq 0.$
2. $a(\pm, 0, 0) > 0, \lim\limits_{x \to \infty} a(\pm, x, 0) < 0.$
3. $b(\pm, 0, y) < 0 \, \forall y \geq 0.$

Assumption 3.2

There is a compact set $D \subset \mathbb{R}^2_+$ such that D is an invariant set for both the systems (3) and (4). Moreover, for all $(x, y) \in \mathbb{R}^2_+$, there is $T \geq 0$ such that $(x_+(t), y_+(t)) \in D$, $(x_-(t), y_-(t)) \in D \; \forall t > T$.

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We can apply a similar argument as above to obtain similar results. Further, we have some advantages in predator-prey type systems. Therefore, we can weeken the hypothesese. Since on the boundary $\{0\} \times \mathbb{R}_+$, we have $\limsup_{t\to\infty} y(t, 0, y_0) = 0$. Therefore, we need only define a value λ, it is given by

$$\lambda = \int_{u^+}^{u^-} (pb(+, u, 0)\mu^+(u) + qb(-, u, 0)\mu^-(u))du. \tag{14}$$

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For any $x_0 > 0, y_0 > 0$.

a) There exists a $\delta_1 > 0$ such that $\limsup_{t\to\infty} x(t, x_0, y_0) \geq \delta_1$, a.s.

b) In case $\lambda > 0$, there exists $\delta > 0$ such that

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a) There exists a $\delta_1 > 0$ such that $\limsup_{t\to\infty} x(t, x_0, y_0) \geq \delta_1$, a.s.

b) In case $\lambda > 0$, there exists $\delta > 0$ such that

$$\limsup_{t\to\infty} y(t, x_0, y_0) > \delta \text{ a.s.}$$

We can apply a similar argument as above to obtain similar results. Further, we have some advantages in predator-prey type systems. Therefore, we can weeken the hypothesese. Since on the boundary $\{0\} \times \mathbb{R}_+$, we have $\limsup_{t\to\infty} y(t, 0, y_0) = 0$. Therefore, we need only define a value λ, it is given by

$$\lambda = \int_{u^+}^{u^-} (pb(+, u, 0)\mu^+(u) + qb(-, u, 0)\mu^-(u))du. \tag{14}$$

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From now on, we suppose that $\lambda > 0$.

Assumption 3.3

One of the following asumption is satisfied:

Case 1: One system is stable and the other is permanent. On the quadrant $\operatorname{int} \mathbb{R}^2_+$, both systems (3), (4) have the globally stable positive states (x^*_+, y^*_+), (x^*_-, y^*_-) respectively.

Case 2: One system is stable and the other is not permanent System (3) has the globally stable positive state (x^*_+, y^*_+). Further, $(u^-, 0)$ is locally stable equilibrium of System (4)

Lemma 3.3

Let Assumption 3.3 be satisfied. Then, for any $\varepsilon > 0$, there exist $\sigma(\varepsilon)$ such that $x^{\pm}(t) > \sigma(\varepsilon), y^{\pm}(t) > \sigma(\varepsilon)$ for all $t > 0$, provided $(x^{\pm}(0), y^{\pm}(0)) \in H_{\varepsilon,M}$.

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Lemma 3.4

Put

$$S' = \left\{ (x, y) = \pi_{t_n}^{\varrho(n)} \cdots \pi_{t_1}^{\varrho(1)} (x_+^*, y_+^*) : 0 < t_1 < t_2 < \cdots < t_n;\ n \in \mathbb{N} \right\}. \tag{15}$$

where $\varrho(k) = (-1)^k$.

Theorem 3.1

a) With probability 1, the closure $\bar{S}$ of S is a subset of the ω-limit set $\Omega(x_0, y_0, \omega)$.

b) If there exists a $t_0 > 0$ such that the point $(\bar{x}_0, \bar{y}_0) = \pi_{t_0}^{-}(x_+^*, y_+^*)$ satisfying the following condition

$$\det \begin{pmatrix} a(+, \bar{x}_0, \bar{y}_0) & a(-, \bar{x}_0, \bar{y}_0) \\ b(+, \bar{x}_0, \bar{y}_0) & b(-, \bar{x}_0, \bar{y}_0) \end{pmatrix} \neq 0, \tag{16}$$

then, with probability 1, the closure $\bar{S}$ of S is the ω-limit set $\Omega(x_0, y_0, \omega)$. Moreover, $\bar{S}$ absorbs all positive solutions in the sense that for any initial value $(x_0, y_0) \in \mathrm{int}\mathbb{R}_+^2$, the value

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Theorem 3.3

Let Hypotheses 3.1, 3.2 and 3.3 be satisfied. If the semigroup $\{P(t)\}_{t\geq 0}$ has an invariant probability measure ν^*, then ν^* has the density f^* and the semigroup is asymptotically stable in the sense that $\lim_{t\to\infty} \|P(t)f - f^*\| = 0 \, \forall f \in \mathcal{D}$.

Applications

- We now apply the results studied in above sections to study the classical predator-prey model

$$\begin{cases} \dot{x}(t) = x(a(\xi_t) - b(\xi_t)x - c(\xi_t)y) \\ \dot{y}(t) = y(-d(\xi_t) + e(\xi_t)x - f(\xi_t)y) \end{cases}, \tag{17}$$

  where $a(\pm), b(\pm), c(\pm), d(\pm), e(\pm), f(\pm)$ are positive constants .

- $$\begin{cases} \dot{x}_+(t) = x_+(a(+) - b(+)x_+ - c(+)y_+) \\ \dot{y}_+(t) = y_+(-d(+) + e(+)x_+ - f(+)y_+), \end{cases} \tag{18}$$

- $$\begin{cases} \dot{x}_-(t) = x_-(a(-) - b(-)x_- - c(-)y_-) \\ \dot{y}_-(t) = y_-(-d(-) + e(-)x_- - f(-)y_-). \end{cases} \tag{19}$$

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The system (18) (resp. (19)) has the equilibrium (x_+^*, y_+^*) (resp. (x_-^*, y_-^*)) where

$$x_\pm^* = \frac{a(\pm)f(\pm) + c(\pm)d(\pm)}{b(\pm)f(\pm) + c(\pm)e(\pm)}, \quad y_\pm^* = \frac{-b(\pm)d(\pm) + a(\pm)e(\pm)}{b(\pm)f(\pm) + c(\pm)e(\pm)}.$$

We choose the domain D limited by the straight lines x $= 0$, $x = x_1$, y $= 0$ and the segment AB, where

$$x_1 = \max\left(\frac{a(+)}{b(+)}, \frac{a(-)}{b(-)}\right), \quad y_1 = \max\left(\frac{2a(+)}{c(+)}, \frac{2a(-)}{c(-)}\right),$$

$$A = (x_1, y_1), B = \left(0, y_1 + \max\left(\frac{2e(+)}{c(+)}, \frac{2e(-)}{c(-)}\right)x_1\right)$$

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Proposition 3.1

We have:

$$\lambda = \left(d(+) - \frac{e(+)}{b(+)}a(+)\right)p + \left(d(-) - \frac{e(-)}{b(-)}a(-)\right)q$$
$$+ \left(\frac{e(+)}{b(+)} - \frac{e(-)}{b(-)}\right)\theta^{-1}\mathrm{sign}\left(\frac{a(+)}{b(+)} - \frac{a(-)}{b(-)}\right)$$
$$\int_{u_1}^{u_2} \frac{|a(+) - b(+)u|^{\frac{\alpha}{a(+)}}|a(-) - b(-)u|^{\frac{\beta}{a(-)}}}{u^{\frac{\alpha}{a(+)}+\frac{\beta}{a(-)}+1}}du.$$

Systems (18) and (19) are globally asymptotically stable

Suppose that

$$\frac{d(i)}{e(i)} < \frac{a(i)}{b(i)} \text{ for } i \in E. \tag{20}$$

In this case, the systems (18) and (19) are globally asymptotically stable with the equilibrium $(x_{\pm}^{*}, y_{\pm}^{*})$.

System (18) is globally asymptotically stable and all positive solutions of System (19) tend to a point on the boundary

Consider the case

$$\frac{d(+)}{e(+)} < \frac{a(+)}{b(+)} \quad \text{but} \quad \frac{d(-)}{e(-)} > \frac{a(-)}{b(-)}.$$

With this assumption, all the positive solutions of system (19) tend either to $(\frac{a(-)}{b(-)}, 0)$

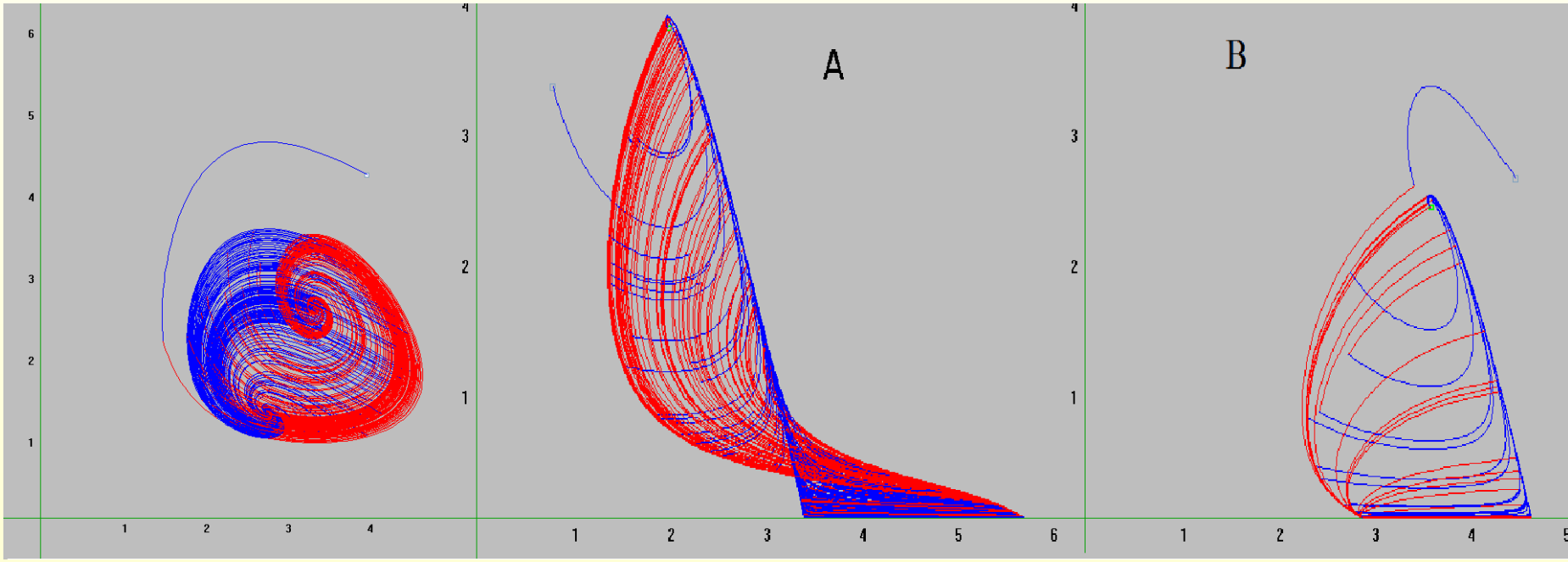
Different to the competion case, in the classical prey-predator model, we can prove the existence of a unique stationary distribution with the support in $\mathrm{int}\mathbb{R}^2_+$

Theorem 3.3

Let Assumption **??** and **??** be satisfied and $\lambda > 0$, (x_t, y_t, ξ_t) has a stationary distribution ν^*. Consequently, if either the hypothesis of Proposition **??** or that of Theorem **??** is satisfied, then ν^* is the unique stationary distribution and ν^* has the density f^* and the semigroup is asymptotically stable in the sense that $\lim_{t\to\infty} \|P(t)f - f^*\| = 0 \, \forall f \in \mathcal{D}$

This part was done with Professor P. Auger, N.T. Hieu 2009.

We give some numerical examples



a classical prey-predator model where none of Systems has asymptotically stable equilibrium

We consider a classical prey-predator model where none of Systems (18) and (19) has asymptotically stable equilibrium, that is $b(\pm) = f(\pm) = 0$. As we know, alll trajectories of (18) and (19) are periodic.

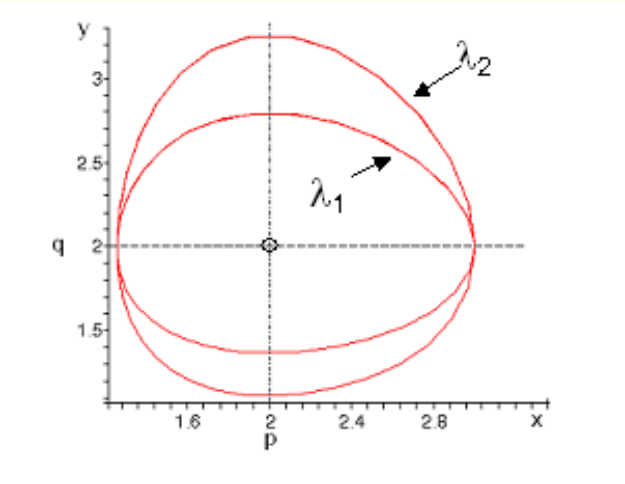


Figure: $a(+) = 6, b(+) = 3, c(+) = 2, d(+) = 12, e(+) = 4, f(+) = 3, a(-) = 12, b(-) = 4, c(-) = 2, d(-) = 9, e(-) = 4, f(-) = 2, x(0) = 3, y(0) = 4, \alpha = 5, \beta = 5$, number of switching $n = 2000$.

Theorem 3.4

Suppose that (**??**) and (**??**) have a common rest point. For any $(x, y) \in \text{int}\mathbb{R}^2_+$,we have with probability 1, either

$$\lim_{t\to\infty}(x(t,x,y), y(t,x,y)) = (p,q), \tag{21}$$

or,

$$\limsup_{t\to\infty} x(t,x,y) = \infty, \liminf_{t\to\infty} x(t,x,y) = 0, \tag{22}$$

$$\limsup_{t\to\infty} y(t,x,y) = \infty, \liminf_{t\to\infty} y(t,x,y) = 0. \tag{23}$$

They are stable but are not asymptotically stable. In this case, the dynamics of (17) is very chaos.

We illustrate our result by simulations. Figure 48 shows the behavior of the trajectory of the systems

$$\begin{cases} \dot{x} = x\left(a(\xi_t) - b(\xi_t)y\right), \\ \dot{y} = y\left(-c(\xi_t) + d(\xi_t)x\right). \end{cases} \tag{24}$$

Case A corresponds to $a(1) = 2$, $b(1) = 2$, $c(1) = 3$, $d(1) = 2$ and $a(2) = 3$, $b(2) = 3$, $c(2) = 6$, $d(2) = 4$ with the initial condition $(2, 1.5)$. In this case, two systems have the rest point in common, the solution of (**??**) turns around the rest point for a while, and then leaves from any compact set.
Case B is related to $a(1) = 2$, $b(1) = 1$, $c(1) = 6$, $d(1) = 3$ and $a(2) = 2$, $b(2) = 2$, $c(2) = 3$, $d(2) = 2$, the initial condition is $(1, 2)$. In this case, two rest points are different, the solution leaves quickly from any compact set.

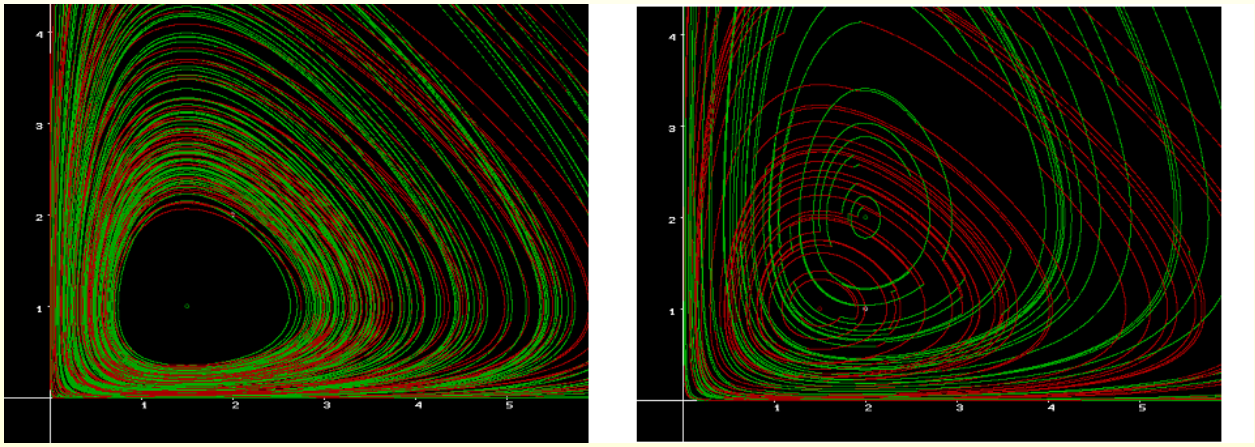


Figure: Left (Case A);Right(Case B)

Conclusion and discussion

This talk provides some results about the asymptotic behavior of a system of two coupled deterministic predator-prey models switching at random. The mathematical analysis presented in this model shows that according to the value of some number λ, one knows the dynamics of solutions of a Kolmogotrov system. From this analysis, one can make suitable predictions about the asymptotic behavior of the overall predator-prey system.

We consider an ecology system where there are two species related by predator-prey relation. Suppose that the evolution of every species depends on the quantity of rainfall for every period. If the rainfall is sufficient (good state), the catch ability of the predator is good and the quantity of every species asymptotes to the positive values (the prey and predator co-exist). Whenever the rainfall is small (bad state), the hunting potential of the predator becomes very weak and the amount of predator gets smaller with increasing of time (the predator vanishes).

Suppose that the rainfall is in a stationary regime (switching stationarily between dry season and rainfall one). If the two states are good, i.e., both $y_1^* > 0$ and $y_2^* > 0$, although the quantity of two species is chaotic, but the system is still permanent. Consequently, none of species is extinct. When there is at least a system having the bad state, i.e., either $y_1^* < 0$ or $y_2^* < 0$ we see that $\liminf_{t\to\infty} y(t) = 0$. Depending on the sign of the value λ, the quantity of the predator $y(t)$ can be recovered or not. In case $\lambda > 0$ we have $\limsup_{t\to\infty} y(t) > 0$, i.e., the amount of the predator is recovered (of course in the rainfall season). If $\lambda < 0$ we have $\lim_{t\to\infty} y(t) = 0$, i.e., the predator vanishes.
However, in reality, when the amount of a species is smaller than a threshold then in fact we consider this species disappears. Thus, the estimate $\liminf_{t\to\infty} y(t) = 0$ tells us that in a predator-prey system developing under the influence of random environment, if there is at least a bad situation, the predator must be vanished in this system. **This conclusion warns us to have a timely decision to protect species in our eco-system.**

Paying attention that even two these systems are in good situation, it does not implies that under the random environmen, our system is permanant.
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THANK YOU

THANK YOU FOR YOUR ATTENTION.